Problems Set By the Herald Lemma Two Dot Four, Reverent Servant of Ylenrith.

Set unto the bright minds of the Empire at the Spring Equinox of Three Hundred and Eighty YE.

The First Problem

There is a room with a hundred tiles in. Each tile is numbered, and the numbers range from one up to a hundred, all different. Each tile also has the rune Ophis on one side, and the rune Diras on the other.

Initially, all the tiles have the Diras side face-up.

A Herald comes in, and flips every tile, and leaves.

Another Herald comes in, and flips the second tile, the fourth tile, the sixth tile, and so on, flipping alternate tiles, up to the hundredth tile, and leaves.

A third Herald comes in, and flips every third tile, starting with the third; and then a fourth comes in, and flips every fourth: and so on, and so on.

Obviously, the last Herald – the hundredth – will come in and flip only the hundredth tile.

At that time, how many tiles show Ophis: and which ones?

The Second Problem

It is the year Three Hundred and Eighty YE.

Recall that the highest common factor of two whole numbers is the greatest number dividing both of them.

If you find the highest common factor of each number between one to three hundred and eighty (inclusive) with three hundred and eighty, and then add all these highest common factors together, what value does this final sum have?

The Third Problem

Three positive whole numbers satisfy the property that the sum of their reciprocals is not less than one. (Recall that the reciprocal of any rational number is the unique other number which, when multiplied by the original number, gives one.)

Determine all such sets of three numbers that satisfy this property.

The Fourth Problem

A whole number is such that when it is multiplied by twenty-four and added to a square of itself, it is sixteen less than a perfect square.

What is the largest value the number could take?

The Fifth Problem

Three positive whole numbers satisfy the property that the sum of their reciprocals is the reciprocal of forty-two. (Recall that the reciprocal of any rational number is the unique other number which, when multiplied by the original number, gives one.)

What is the largest value any of these numbers can take?

The Sixth Problem

It is the year Three Hundred and Eighty YE.

Consider, for each number dividing three hundred and eighty, the number of factors that the square of each of these divisors has. If you add each of these numbers of factors in turn, you will find the sum of the number of factors of the square of each number which divides three hundred and eighty.

Show that this sum is the same as the square of the number of factors of three hundred and eighty.

The Seventh Problem

Recall the room with the tiles, which have Ophis and Diras on either side, from the First Problem.

One day, the same procedure as before is started: heralds come in, turn some tiles – for instance, the fifth herald flips every fifth tile before leaving.

However, on this day some Heralds decide to neglect their duties, and when they enter the room they do not flip any tiles at all, and leave after an appropriate amount of time. The next Herald is unaware of their fellow's failure, and so flips the tiles they would have flipped anyway. For instance, if the seventh Herald does not do their job, then nobody flips every seventh tile, and then the eighth Herald flips every eighth tile, etc.

After all the Heralds have left, only the first tile shows Ophis, and the rest show Diras.

Characterize and count the absent Heralds.

Lemma Two Dot Four, Servant of Ylenrith, shall return on the Sunday of the Spring Equinox, in the year Three Hundred and Eighty YE, at the hour of three past midday, in order to discuss these problems and talk through any solutions. If you could bring a writing implement and something to write with, that would be ideal.

Lemma Two Dot Four is excited to meet those who share its passion for the purity of numbers!